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# $h$-Fuzzy Quantum Logics

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Families of fuzzy subsets equipped by continuous fuzzy connectives which are quantum logics in a traditional sense are studied. As a special case, we obtain a generalized fuzzy quantum logic introduced recently by Pykacz.

## 1. INTRODUCTION

The quantum logic axiomatic approach to the foundations of quantum mechanics studied, for example, by Varadarajan (1968) is based on the following notion.

*Definition 1.* A quantum logic $\mathscr{L}$ is an orthomodular $\sigma$-orthocomplete orthoposet, i.e., a partially ordered set $\mathscr{L}$ containing the smallest element **0** and the greatest element **1** equipped with an orthocomplementation $\perp: \mathscr{L} \to \mathscr{L}$ such that the following conditions are fulfilled:

(i) $(a^\perp)^\perp = a$ for any $a \in \mathscr{L}$ (law of repeated negation).

(ii) If $a, b \in \mathscr{L}$, $a \leq b$, then $b^\perp \leq a^\perp$ (order reversing).

(iii) For any $a \in \mathscr{L}$, $a \wedge a^\perp = \mathbf{0}$ (law of contradiction) and $a \vee a^\perp = \mathbf{1}$ (excluded middle law), where $\wedge$ is the meet (the greatest lower bound in $\mathscr{L}$) and $\vee$ is the join (the least upper bound in $\mathscr{L}$).

(iv) If $a, b \in \mathscr{L}$, $a \leq b$, then there is an element $c \in \mathscr{L}$, $c \leq a^\perp$, such that $b = a \vee c$ (orthomodular law), where $c = a^\perp \wedge b = (a \vee b^\perp)^\perp$.

(v) If $\{a_n\} \subset \mathscr{L}$, $a_n \perp a_m$ (i.e., $a_n \leq a_m^\perp$) whenever $n \neq m$, then the join $\bigvee a_n$ exists in $\mathscr{L}$. ■

Let $\mathscr{U} \neq \emptyset$ be a given universe. We denote by $\mathscr{F}(\mathscr{U})$ the system of all fuzzy subsets of $\mathscr{U}$, i.e., $\mathscr{F}(\mathscr{U}) = [0, 1]^{\mathscr{U}}$. Recall that a fuzzy subset

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$A \in \mathcal{F}(\mathcal{U})$ is a mapping $A: \mathcal{U} \to [0, 1]$. Crisp subsets of $\mathcal{U}$ are identified with their characteristic functions, i.e., $A$ is crisp iff $A$ maps $\mathcal{U}$ into $\{0, 1\}$. The partial ordering on $\mathcal{F}(\mathcal{U})$ is induced by the natural ordering on [0, 1], i.e., $A \le B$ iff $A(u) \le B(u)$ for any $u \in \mathcal{U}$. Fuzzy connectives of fuzzy complementation, fuzzy union, and fuzzy intersection, respectively, are supposed to be induced pointwisely by continuous functions $\mathbf{c}$ (one-place function, a negation on [0, 1]), $\mathbf{S}$ (two-place function, a $t$-conorm on [0, 1]), and $\mathbf{T}$ (two-place function, a $t$-norm on [0, 1]), respectively. Hence we have $A^c(u) = \mathbf{c}(A(u))$, $(A \cup B)(u) = \mathbf{S}(A(u), B(u))$, and $(A \cap B)(u) = \mathbf{T}(A(u), B(u))$ for any $u \in \mathcal{U}$. Further, $\mathbf{S}$ and $\mathbf{T}$ are supposed to be $\mathbf{c}$-dual pair, i.e., $\mathbf{S}(x, y) = \mathbf{c}^{-1}(\mathbf{T}(\mathbf{c}(x), \mathbf{c}(y)))$, $x, y \in [0, 1]$. For more details see, e.g., Dubois and Prade (1985).

Let $\mathcal{V}$ be any subfamily of $\mathcal{F}(\mathcal{U})$, $\mathcal{V} \subset \mathcal{F}(\mathcal{U})$. The join $\vee$ and the meet $\wedge$ in $\mathcal{V}$ are induced by the partial ordering of fuzzy subsets. Recall that if $\mathcal{V} = \mathcal{F}(\mathcal{U})$, then the join $\vee$ and the meet $\wedge$ coincide with the original fuzzy union and fuzzy intersection (Zadeh, 1965), respectively, which are induced by the $t$-conorm $\mathbf{S}_0$, $\mathbf{S}_0(x, y) = \max(x, y)$ and by the $t$-norm $\mathbf{T}_0$, $\mathbf{T}_0(x, y) = \min(x, y)$, respectively. In general, the join $\vee$ and the meet $\wedge$ in $\mathcal{V}$ need not coincide with Zadeh fuzzy connectives.

The aim of this paper is to study the following problem: when is a system of fuzzy subsets $\mathcal{V} \subset \mathcal{F}(\mathcal{U})$ equipped by the fuzzy connectives from $\mathcal{F}(\mathcal{U})$ (and hence closed under these connectives) a quantum logic in the sense of Varadarajan (1968) such that the fuzzy complementation corresponds to the orthocomplementation and the fuzzy union (intersection) corresponds to the join (meet)? One example was given recently by Pykacz (1994), who solved this problem in the case of the original (Zadeh, 1965) fuzzy complementation, i.e., $\mathbf{c}(x) = 1 - x$ on [0, 1] and the Giles (1976) bold union [i.e., $\mathbf{S}(x, y) = \min(x + y, 1)$] and bold intersection [i.e., $\mathbf{T}(x, y) = \max(x + y - 1, 0)$]. We study the proposed problem from a more general point of view.

## 2. QUANTUM LOGIC AXIOMS AND FUZZY CONNECTIVES

The quantum logic axioms (i) and (ii) are fulfilled on $\mathcal{F}(\mathcal{U})$ (putting $A^{\perp} = A^c$) if and only if the complementation $\mathbf{c}$ is an order-reversing involution on [0, 1]. But then $\mathbf{c}$ is generated by a generator $g$, $g: [0, 1] \to [0, 1]$, $g(0) = 0$, $g(1) = 1$, $g$ is continuous and strictly increasing,

$$\forall x \in [0, 1]: \quad \mathbf{c}(x) = g^{-1}(1 - g(x)) \tag{1}$$

For more details see, e.g., Trillas (1979) or Dubois and Prade (1985). In what follows, we will suppose that $\mathbf{c}$ is defined by equation (1). Now, it is easy to see that any system $\mathcal{V} \subset \mathcal{F}(\mathcal{U})$ closed under fuzzy complementation fulfills the axioms (i) and (ii).

Let fuzzy union $\cup$ on $\mathscr{F}(\mathscr{U})$ be induced by a continuous $t$-conorm **S**. Using the results of Ling (1965) (see also Schweizer and Sklar, 1983), we know that **S** is an ordinal sum of Archimedean $t$-conorms $\{\mathbf{S}_i\}_{i\in I}$, i.e., there is a disjoint family $\{]a_i, b_i[,\ i\in I\}$ of open subintervals of $[0, 1]$ so that

$$\mathbf{S}(x, y) = a_i + (\mathbf{S}_i((x - a_i)/(b_i - a_i), (y - a_i)/(b_i - a_i))) \cdot (b_i - a_i)$$

if both $x$ and $y$ are contained in some $]a_i, b_i[$, $i\in I$, and $\mathbf{S}(x, y) = \max(x, y)$ otherwise. Recall that a continuous $t$-conorm **S** is called Archimedean if for any $x\in]0, 1[$ it is $\mathbf{S}(x, x) > x$. By Ling (1965), any Archimedean $t$-conorm **S** is generated by an additive generator $h$, $h\colon [0, 1]\to[0, \infty]$, $h(0) = 0$, $h$ is continuous and strictly increasing,

$$\forall x, y\in[0, 1]\colon \quad \mathbf{S}(x, y) = h^*(h(x) + h(y)) \tag{2}$$

where $h^*$ is a pseudoinverse of $h$ defined through

$$h^*(t) = h^{-1}(\min(h(1), t)), \qquad t\in[0, \infty] \tag{3}$$

If $h(1) = \infty$, then $h^* = h^{-1}$ and the corresponding $t$-conorm **S** is called a strict $t$-conorm. If $h(1) < \infty$, then we can suppose without any loss of generality that $h(1) = 1$ and the corresponding $t$-conorm **S** is called a nilpotent $t$-conorm. Taking into account the axiom (iv), the De Morgan law should hold. Consequently the $t$-norm **T** inducing the fuzzy intersection $\cap$ is given by

$$\mathbf{T}(x, y) = \mathbf{c}(\mathbf{S}(\mathbf{c}(x), \mathbf{c}(y))) = f^*(f(x) + f(y))$$

where $f(x) = h(\mathbf{c}(x))$ and $f^*(t) = f^{-1}(\min(f(0), t))$.

Now, the excluded middle law is fulfilled on $\mathscr{F}(\mathscr{U})$ (replacing the join by the fuzzy union) if and only if

$$\mathbf{S}(x, \mathbf{c}(x)) = 1 \qquad \text{for any} \quad x\in[0, 1] \tag{4}$$

But then the cardinality of the index set $I$ in the ordinal sum representation of **S** should be 1 and the corresponding open interval $]a_1, b_1[$ is equal to $]0, 1[$. If not, then there would be an element $x\in]0, 1[$ such that $\mathbf{S}(x, \mathbf{c}(x)) = \max(x, \mathbf{c}(x)) < 1$, which is a contradiction with equation (4). Hence **S** is generated by an additive generator $h$ through equation (2). Further, for any strict $t$-conorm **S** one has $\mathbf{S}(x, y) = 1$ iff $\max(x, y) = 1$, i.e., $\mathbf{S}(x, \mathbf{c}(x)) = 1$ only for $x\in\{0, 1\}$. Hence **S** should be a nilpotent $t$-conorm and we can suppose that its additive generator $h$ fulfills $h(1) = 1$, i.e.,

$$\mathbf{S}(x, y) = h^{-1}(\min(h(x) + h(y), 1)), \qquad x, y\in[0, 1] \tag{5}$$

Then $\mathbf{S}(x, \mathbf{c}(x)) = 1$ iff $h(x) + h(\mathbf{c}(x)) \geq 1$. Further, the law of contradiction is true on $\mathscr{F}(\mathscr{U})$ (replacing the meet by the fuzzy intersection) if and only if the excluded middle law is true. A simple computation shows that the axiom (iii) is fulfilled on $\mathscr{F}(\mathscr{U})$ iff

$$\mathbf{c}(x) \geq h^{-1}(1 - h(x)) = \mathbf{d}(x) \qquad \text{for any} \quad x \in [0, 1] \tag{6}$$

where $\mathbf{d}$ is a fuzzy complementation operator generated by $h$. Note that (6) ensures the validity of the fuzzy analog of the orthomodular law on $\mathscr{F}(\mathscr{U})$, i.e., whenever $A, B \in \mathscr{F}(\mathscr{U})$, $A \leq B$, then there is $C \in \mathscr{F}(\mathscr{U})$, $C \leq A^c$, such that $B = A \cup C$. Hence the fuzzy analog of the axiom (iv) would hold on $\mathscr{F}(\mathscr{U})$ if the equalities $C = A^c \cap B = (A \cup B^c)^c$ hold. The last equality is obvious due to the duality of $\mathbf{T}$ and $\mathbf{S}$ inducing the fuzzy connectives of intersection and union. Because of pointwise definition of fuzzy connectives through $\mathbf{c}$, $\mathbf{S}$, and $\mathbf{T}$, respectively, it is enough to prove the validity of the following statement:

$$\text{for any} \quad 0 \leq a \leq b \leq 1 \quad \text{one has}$$

$$b = \mathbf{S}(a, \mathbf{c}(\mathbf{S}(a, \mathbf{c}(b)))) \tag{7}$$

*Lemma 1.* Equation (7) is true if and only if both $\mathbf{c}$ and $\mathbf{S}$ have the same generator $h$, i.e., if $\mathbf{c}(x) = h^{-1}(1 - h(x))$ and $\mathbf{S}(x, y) = h^{-1}(\min(1, h(x) + h(y)))$. Then the corresponding $t$-norm $\mathbf{T}$ has a generator $f = 1 - h$.

*Proof.* Fix an element $a < 1$ and let $b_n = 1 - (1 - a)/n$. Then $b_n = \mathbf{S}(\mathbf{a}, \mathbf{c}(\mathbf{S}(a, \mathbf{c}(b_n)))) < 1$, which implies $h(b_n) = h(a) + h(\mathbf{c}(\mathbf{S}(a, \mathbf{c}(b_n))))$, $n = 1, 2, \ldots$. From the continuity of $\mathbf{c}$ and $\mathbf{S}$ one has

$$1 = \lim h(b_n) = \lim(h(a) + h(\mathbf{c}(\mathbf{S}(a, \mathbf{c}(b_n))))) = h(a) + h(\mathbf{c}(a))$$

i.e., $\mathbf{c}(a) = h^{-1}(1 - h(a))$. For $a = 1$, it is obvious that $\mathbf{c}(1) = 0 = h^{-1}(1 - 1)$. We have just shown that $\mathbf{c}$ is generated by the same generator $h$ as the $t$-conorm $\mathbf{S}$ is—this is a necessary condition for the validity of equation (7).

Now, we show the sufficiency. Let $h$ be a common generator for both $\mathbf{c}$ and $\mathbf{S}$. For $0 \leq a \leq b \leq 1$, we have

$$\begin{aligned}
&\mathbf{S}(a, \mathbf{c}(\mathbf{S}(a, \mathbf{c}(b)))) \\
&\quad = h^*[h(a) + h^{-1}(1 - h(h^*[h(a) + h(h^{-1}(1 - h(b)))]))] \\
&\quad = h^*[h(a) + h^{-1}(1 - h(h^*[h(a) + 1 - h(b)]))] \\
&\quad = h^*[h(a) + 1 - (h(a) + 1 - h(b))] \\
&\quad = h^*[h(b)] = b
\end{aligned}$$

which is equation (7), in fact.

From the duality of **T** and **S** we have

$$f(x) = h(\mathbf{c}(x)) = h(h^{-1}(1 - h(x))) = 1 - h(x)$$

for any $x \in [0, 1]$. ■

The validity of the fuzzy analog of the axiom (*v*) on $\mathscr{F}(\mathscr{U})$ is obvious. All the previous facts are summarized in the following theorem.

*Theorem 1.* The system $\mathscr{F}(\mathscr{U})$ of all fuzzy subsets of a universe $\mathscr{U}$ fulfills the fuzzy analogs of the axioms (i)–(v) of quantum logic using the fuzzy complementation as an orthocomplement, the fuzzy union as the join, and the fuzzy intersection as the meet, respectively, if and only if the fuzzy connectives on $\mathscr{F}(\mathscr{U})$ are induced by a complementation **c** and by a $t$-conorm **S** with a common generator $h$ and by a $t$-norm **T** with generator $f = 1 - h$. ■

*Example 1.* Take a generator $h(x) = x$, $x \in [0, 1]$. Then the corresponding complementation operator $c(x) = 1 - x$ induces the Zadeh fuzzy complementation. The corresponding $t$-conorm $\mathbf{S}(x, y) = \min(1, x + y)$ induces the Giles bold union and its dual $t$-norm $\mathbf{T}(x, y) = \max(0, x + y - 1)$ induces the Giles bold intersection. This is the background of Pykacz' (1994) approach to fuzzy quantum logics. ■

## 3. *h*-FUZZY QUANTUM LOGICS

Let $h$ be a generator of fuzzy connectives on $\mathscr{F}(\mathscr{U})$, $h(1) = 1$. Let $\mathscr{V}$ be any nonempty subsystem of $\mathscr{F}(\mathscr{U})$ closed under fuzzy complementation and under countable fuzzy unions. It is evident that the axioms (i)–(v) are true also on $\mathscr{V}$. Further, $\mathscr{V}$ contains the smallest element $\varnothing$ and the greatest element $\mathscr{U}$ (take $A \in \mathscr{V}$; then $\varnothing = A \cap A^c \in \mathscr{V}$ and $\mathscr{U} = A \cup A^c \in \mathscr{V}$). Why may $\mathscr{V}$ not be a quantum logic in general? This is caused by the possible noncompatibility between the ordering and the fuzzy connectives. For $\mathscr{V}$ to be a quantum logic, it is necessary to exclude, for example, any nontrivial element orthogonal with itself.

*Definition 2.* Let $h$ be a generator, $h(1) = 1$. A fuzzy subset $A \in \mathscr{F}(\mathscr{U})$ is called an $h$-weak empty set if $A \perp A$, i.e., $A \leq A^{\perp}$, where we put $A^{\perp}(u) = h^{-1}(1 - h(A(u)))$, $u \in \mathscr{U}$.

It is easy to see that $A$ is an $h$-weak empty set iff $A \leq h^{-1}(1/2)$.

*Theorem 2.* Let the fuzzy complementation and the fuzzy union on $\mathscr{F}(\mathscr{U})$ be induced by operators **c** and **S** generated by a common generator $h$, $h(1) = 1$. Let $\varnothing \neq \mathscr{V} \subset \mathscr{F}(\mathscr{U})$ be closed under fuzzy complementation and under countable fuzzy unions of pairwise orthogonal elements. Then the fuzzy union of sequences of pairwise orthogonal elements from $\mathscr{V}$ is the same as the join with respect to the natural ordering on $\mathscr{V}$ if and only if the only $h$-weak empty set in $\mathscr{V}$ is $\varnothing$.

*Proof.* If an $h$-weak empty set $A \neq \varnothing$ is contained in $\mathscr{V}$, then $A \perp A$. It is evident that $A \neq \mathscr{U}$. Hence there is $u \in \mathscr{U}$ such that $A(u) \in ]0, 1[$. But then $S(A(u), A(u)) > A(u)$, i.e., $A \cup A \neq A = A \vee A$.

Now, let $A, B \in \mathscr{V}$, $A \perp B$, and let $A \cup B$ be not the lowest upper bound of $A$ and $B$ in $\mathscr{V}$. Then there is an upper bound $C \in \mathscr{V}$ of $A$ and $B$ such that either $A \cup B \geq C$ and there is an element $u^* \in U$ such that $S(A(u^*), B(u^*)) > C(u)$, or $C$ is not comparable with $A \cup B$. In the first case, put $D = C \cup (A \cup B)^{\perp}$. Then $D \in \mathscr{V}$. For any $u \in \mathscr{U}$, one has

$$h(D(u)) = \min(1, h(C(u)) + 1 - h((A \cup B)(u))))$$

$$= 1 + h(C(u)) - h(A(u)) - h(B(u))$$

$$\geq 1 - \min(h(A(u)), h(B(u))) \geq 1/2$$

The last inequality follows from the orthogonality of $A$ and $B$: $A \leq B^{\perp}$, i.e., $A(u) \leq h^{-1}(1 - h(B(u)))$ for any $u \in \mathscr{U}$; consequently $h(A(u)) \leq 1 - h(B(u))$, which implies the result. Further, it is evident that $D(u^*) < 1$. For $D^{\perp}$, we see that $D^{\perp}(u) = h^{-1}(1 - h(D(u))) \leq h^{-1}(1/2)$, i.e., $D^{\perp}$ is an $h$-weak empty set. Further, $D^{\perp} \in \mathscr{V}$ because of $D \in \mathscr{V}$ and $D^{\perp} \neq \varnothing$ because of $D^{\perp}(u^*) > 0$.

Now, let $C$ be an upper bound of both $A$ and $B$ which is not comparable with $A \cup B$. For the sake of simplicity, in this step we suppose $h(x) = i(x) = x$ on $[0, 1]$ [i.e., we suppose the original approach from Pykacz (1994)]. In case of a general generator $h$, it is enough to use $h(A(u))$ instead of $A(u)$ and similarly for $B(u)$, $C(u)$, $D(u)$, etc., in what follows. Put

$$\mathscr{U}_1 = \{u \in \mathscr{U};\, C(u) < A(u) + B(u)\}$$

$$\mathscr{U}_2 = \{u \in \mathscr{U};\, C(u) = A(u) + B(u)\}$$

$$\mathscr{U}_3 = \{u \in \mathscr{U};\, C(u) > A(u) + B(u)\}$$

The noncomparability of $C$ and $A \cup B$ ($= A + B$ because of $A \perp B$) is equivalent with the nonemptiness of both $\mathscr{U}_1$ and $\mathscr{U}_3$. The pairwise orthogonality of $A$, $B$, and $C^{\perp}$ ensures $D = A \cup B \cup C^{\perp} \in \mathscr{V}$. We have

$$D(u) = \min(1, A(u) + B(u) + 1 - C(u))$$

$$= \begin{cases} 1 & \text{on } \mathscr{U}_1 \cup \mathscr{U}_2 \\ A(u) + B(u) + 1 - C(u) < 1 & \text{on } \mathscr{U}_3 \end{cases}$$

$D$ is again an upper bound of both $A$ and $B$ and hence $A$, $B$, and $D^\perp$ are pairwise orthogonal and consequently $E = A \cup B \cup D^\perp \in \mathscr{V}$. We have

$$E(u) = \min(1, A(u) + B(u) + 1 - D(u)) = \begin{cases} A(u) + B(u) & \text{on } \mathscr{U}_1 \cup \mathscr{U}_2 \\ C(u) & \text{on } \mathscr{U}_3 \end{cases}$$

We have $C \leq E$, i.e., $C \perp E^\perp$. Then $F = C \cup E^\perp \in \mathscr{V}$ and hence $F^\perp = C^\perp \cap E \in \mathscr{V}$, where

$$F^\perp(u) = \max(0, E(u) - C(u)) = \begin{cases} A(u) + B(u) - C(u) & \text{on } \mathscr{U}_1 \cup \mathscr{U}_2 \\ 0 & \text{on } \mathscr{U}_3 \end{cases}$$

It is evident that $0 < F^\perp(u) < \min(A(u), B(u)) \leq 1/2$ on $\mathscr{U}_1$ and $F^\perp(u) = 0$ otherwise, i.e., $F^\perp$ is an $i$-weak empty set different from $\varnothing$ contained in $\mathscr{V}$.

We have just shown that if the only $h$-weak empty set contained in $\mathscr{V}$ is $\varnothing$, then for any couple $A, B$ of orthogonal elements from $\mathscr{V}$ one has

$$A \cup B = A \vee B$$

where $A \vee B$ is the join of elements of $\mathscr{V}$ induced by the natural ordering on $\mathscr{V}$.

Let $\{A_n\}_{n=1}^{\infty} \subset \mathscr{V}$ be a sequence of pairwise orthogonal elements from $\mathscr{V}$. Put

$$B_n = \bigcup_{i=1}^{n} A_i, \quad n = 1, 2, \ldots, \qquad \text{and} \qquad D = \bigcup_{i=1}^{\infty} A_i$$

Then $B_n$, $n \in \mathbf{N}$, and $D$ are contained in $\mathscr{V}$, the sequence $\{B_n\}_{n=1}^{\infty}$ is nondecreasing, and $D$ is its least upper bound in $\mathscr{V}$,

$$D = \bigvee_{n=1}^{\infty} B_n$$

We show by induction that $B_n = \bigvee_{i=1}^{n} A_i$. For $n = 2$, it was shown above. Let us suppose that the foregoing equality is true for some $n \in \mathbf{N}$. Then the pairwise orthogonality of the sequence $\{A_n\}$ implies $A_i \leq A_{n+1}^\perp$ for all $i = 1, \ldots, n$, and consequently

$$B_n = \bigvee_{i=1}^{n} A_i \leq A_{n+1}^\perp, \qquad \text{i.e.,} \quad B_n \perp A_{n+1}$$

Both $B_n$ and $A_{n+1}$ are contained in $\mathscr{V}$, they are orthogonal, and hence

$$B_{n+1} = B_n \cup A_{n+1} = B_n \vee A_{n+1} = \bigvee_{i=1}^{n} A_i \vee A_{n+1} = \bigvee_{i=1}^{n+1} A_i$$

We have just shown that the fuzzy union of any finite sequence of pairwise orthogonal elements from $\mathscr{V}$ coincides with its join in $\mathscr{V}$. The following equality completes the proof of the theorem:

$$\bigcup_{n=1}^{\infty} A_n = D = \bigvee_{n=1}^{\infty} B_n = \bigvee_{n=1}^{\infty} \left( \bigvee_{i=1}^{n} A_i \right) = \bigvee_{n=1}^{\infty} A_n \quad \blacksquare$$

The foregoing results justify the following definition.

*Definition 3.* Let $h$ be a generator, $h$: $[0, 1] \to [0, 1]$, $h(0) = 0$, $h(1) = 1$, $h$ is continuous and strictly increasing (then $h$ is called a normed generator). A nonempty family $\mathscr{V} \subset \mathscr{F}(\mathscr{U})$ of fuzzy subsets of universe $\mathscr{U}$ will be called an $h$-fuzzy quantum logic if it satisfies the following properties:

1. $\mathscr{V}$ is closed under $h$-fuzzy complementation, i.e., if $A \in \mathscr{V}$, then $A^{\perp} \in \mathscr{V}$, where $A^{\perp}(u) = h^{-1}(1 - h(A(u)))$, $u \in \mathscr{U}$.
2. $\mathscr{V}$ is closed under countable $h$-fuzzy unions of pairwise orthogonal elements, i.e., if $\{A_n\} \subset \mathscr{V}$, $A_n \leq A_m^{\perp}$ for $n \neq m$; then $\bigcup A_n \in \mathscr{V}$, where $(\bigcup A_n)(u) = h^{-1}(\min(1, \sum h(A_n(u))))$, $u \in \mathscr{U}$.
3. $\varnothing$ is the only $h$-weak empty set contained in $\mathscr{V}$.

*Remark 1.* Let $i$ be the identity on $[0, 1]$, $i(x) = x$. Then $i$ is a normed generator and any $i$-fuzzy quantum logic is a generalized fuzzy quantum logic of Pykacz (1994).

*Lemma 2.* Let $h$ and $g$ be two normed generators. Let $\mathscr{V} \subset \mathscr{F}(\mathscr{U})$ be an $h$-fuzzy quantum logic. Then $\mathscr{T} = g^{-1}(h(\mathscr{V})) = \{B \in \mathscr{F}(\mathscr{U}), \exists A \in \mathscr{V}$, for any $u \in \mathscr{U}$: $B(u) = g^{-1}(h(A(u)))\}$ is a $g$-fuzzy quantum logic.

*Proof.* For the transformation of $h$-fuzzy connectives into $g$-fuzzy connectives see, e.g., Mesiar (1992). We will show only that if $A$ is an $h$-weak empty set, then $B = g^{-1}(h(A))$ is a $g$-weak empty set. Recall that $A$ is an $h$-weak empty set iff $A(u) \leq h^{-1}(1/2)$ for any $u \in \mathscr{U}$. Both $g^{-1}$ and $h$ are strictly increasing and hence $g^{-1}(h(A(u))) \leq g^{-1}(h(h^{-1}(1/2))) = g^{-1}(1/2)$, $u \in \mathscr{U}$, which implies that $B$ is a $g$-weak empty set. $\blacksquare$

Due to the previous lemma, for any normed generator $h$, the system of all $h$-fuzzy quantum logics is isomorphic to the system of all generalized fuzzy quantum logics of Pykacz. Using this isomorphism (or directly from the Definition 3 and our previous results), we get the following corollary.

*Corollary 1.* Let $h$ be any normed generator and $\mathscr{V}$ an $h$-fuzzy quantum logic. Then $\mathscr{V}$ is a quantum logic, i.e., an orthomodular $\sigma$-orthocomplete orthoposet with respect to the standard fuzzy set inclusion as partial order and the $h$-fuzzy complementation as orthocomplementation. For pairwise orthogonal elements the join coincides with the $h$-fuzzy union.

*Remark 2.* In the early version of Theorem 1 in Pykacz (1994) the existence of a join of any sequence of pairwise orthogonal elements from $\mathcal{V}$ (with respect to the fuzzy ordering) was required, where $\mathcal{V}$ is a generalized fuzzy quantum logic (i.e., an $i$-fuzzy logic), for ensuring $\mathcal{V}$ be a traditional quantum logic. Due to our Theorem 2, this requirement is superfluous.

We think that the $h$-fuzzy quantum logics are the only families of fuzzy subsets which are quantum logics in the classical sense of Varadarajan (1968).

*Hypothesis.* Let a nonempty family $\mathcal{V} \subset \mathcal{F}(\mathcal{U})$ equipped with pointwisely generated fuzzy connectives be a quantum logic. Then there is a normed generator $h$ so that $\mathcal{V}$ is an $h$-fuzzy quantum logic.

Note that if the above hypothesis is true, then the only fuzzy subset systems (equipped with pointwisely generated fuzzy connectives) which are traditional quantum logics are (up to an isomorphism) the generalized fuzzy quantum logics of Pykacz.

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